

UNRAMIFIED BRAUER GROUPS AND ISOCLINISM

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ABSTRACT. We show that if G_1 and G_2 are isoclinic groups, then their Bogomolov multipliers are isomorphic.

1. INTRODUCTION

Let G be a finite group and V a faithful representation of G over \mathbb{C} . Then there is a natural action of G upon the field of rational functions $\mathbb{C}(V)$. Noether's problem [9] asks as to whether the field of G -invariant functions $\mathbb{C}(V)^G$ is purely transcendental over \mathbb{C} , i.e., whether the quotient space V/G is *rational*. A question related to the above mentioned is whether V/G is *stably rational*, that is, whether there exist independent variables x_1, \dots, x_r such that $\mathbb{C}(V)^G(x_1, \dots, x_r)$ becomes a pure transcendental extension of \mathbb{C} . This problem has close connection with Lüroth's problem [10] and the inverse Galois problem [12, 11]. By Hilbert's Theorem 90 stable rationality of V/G does not depend upon the choice of V , but only on the group G . Saltman [11] found examples of groups G of order p^9 such that V/G is not stably rational over \mathbb{C} . His main method was application of the unramified cohomology group $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over \mathbb{C} that were not rational. Bogomolov [2] further explored this cohomology group. He proved that $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$(1.0.1) \quad B_0(G) = \bigcap_{\substack{A \leq G, \\ A \text{ abelian}}} \ker \text{res}_A^G,$$

where $\text{res}_A^G : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map. The group $B_0(G)$ is a subgroup of the *Schur multiplier* $H^2(G, \mathbb{Q}/\mathbb{Z})$ of G . Kunyavskii [5] coined the term the *Bogomolov multiplier* of G for the group $B_0(G)$.

We recently proved [7] that $B_0(G)$ is naturally isomorphic to $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where $\tilde{B}_0(G)$ is the kernel of the commutator map $G \wedge G \rightarrow [G, G]$, and $G \wedge G$ is a quotient of the *non-abelian exterior square* of G (see Section 2 for further details). This description of $B_0(G)$ is purely combinatorial, and allows for efficient computations of $B_0(G)$, and a Hopf formula for $B_0(G)$. We also note here that the group $\tilde{B}_0(G)$ can be defined for any (possibly infinite) group G .

Recently, Hoshi, Kang, and Kunyavskii [4] classified all groups of order p^5 with non-trivial Bogomolov multiplier; another classification was found in [8]. It turns out that only examples of such groups appear within the same isoclinism family, where isoclinism is the notion defined by P. Hall [3]. The following question was posed in [4]:

Question 1.1 ([4]). *Let G_1 and G_2 be isoclinic p -groups. Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, or at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?*

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The purpose of this note is to answer the second part of the above question in the affirmative:

Theorem 1.2. *Let G_1 and G_2 be isoclinic groups. Then $\tilde{B}_0(G_1) \cong \tilde{B}_0(G_2)$. In particular, if G_1 and G_2 are finite, then $B_0(G_1)$ is isomorphic to $B_0(G_2)$.*

2. PROOF OF THEOREM 1.2

We first recall the definition of $G \wr G$ from [7]. For $x, y \in G$ we write ${}^x y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$. Let G be any group (possibly infinite). We form the group $G \wr G$, generated by the symbols $m \wr n$, where $m, n \in G$, subject to the following relations:

$$(2.0.1) \quad \begin{aligned} mm' \wr n &= ({}^m m' \wr {}^m n)(m \wr n), \\ m \wr nn' &= (m \wr n)({}^n m \wr {}^n n'), \\ x \wr y &= 1, \end{aligned}$$

for all $m, m', n, n' \in G$, and all $x, y \in G$ with $[x, y] = 1$. The group $G \wr G$ is a quotient of the non-abelian exterior square $G \wedge G$ of G defined by Miller [6]. There is a surjective homomorphism $\kappa : G \wr G \rightarrow [G, G]$ defined by $\kappa(x \wr y) = [x, y]$ for all $x, y \in G$. Denote $\tilde{B}_0(G) = \ker \kappa$. By [7] we have the following:

Theorem 2.1 ([7]). *Let G be a finite group. Then $B_0(G)$ is naturally isomorphic to $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, and thus $B_0(G) \cong \tilde{B}_0(G)$.*

Let L be a group. A function $\phi : G \times G \rightarrow L$ is called a \tilde{B}_0 -pairing if for all $m, m', n, n' \in G$, and for all $x, y \in G$ with $[x, y] = 1$,

$$\begin{aligned} \phi(mm', n) &= \phi({}^m m', {}^m n)\phi(m, n), \\ \phi(m, nn') &= \phi(m, n)\phi({}^n m, {}^n n'), \\ \phi(x, y) &= 1. \end{aligned}$$

Clearly a \tilde{B}_0 -pairing ϕ determines a unique homomorphism of groups $\phi^* : G \wr G \rightarrow L$ such that $\phi^*(m \wr n) = \phi(m, n)$ for all $m, n \in G$.

We now turn to the proof of Theorem 1.2. Let G_1 and G_2 be isoclinic groups, and denote $Z_1 = Z(G_1)$, $Z_2 = Z(G_2)$. By definition [3], there exist isomorphisms $\alpha : G_1/Z_1 \rightarrow G_2/Z_2$ and $\beta : [G_1, G_1] \rightarrow [G_2, G_2]$ such that if $\alpha(a_1 Z_1) = a_2 Z_2$ and $\alpha(b_1 Z_1) = b_2 Z_2$, then $\beta([a_1, b_1]) = [a_2, b_2]$ for all $a_1, b_1 \in G_1$. Define a map $\phi : G_1 \times G_1 \rightarrow G_2 \wr G_2$ by $\phi(a_1, b_1) = a_2 \wr b_2$, where a_i, b_i are as above. To see that this is well defined, suppose that $\alpha(a_1 Z_1) = a_2 Z_2 = \bar{a}_2 Z_2$ and $\alpha(b_1 Z_1) = b_2 Z_2 = \bar{b}_2 Z_2$. Then we can write $\bar{a}_2 = a_2 z$ and $\bar{b}_2 = b_2 w$ for some $w, z \in Z_2$. By definition of $G_2 \wr G_2$ we have that $\bar{a}_2 \wr \bar{b}_2 = a_2 \wr b_2$, hence ϕ is well defined.

Suppose that $a_1, b_1 \in G_1$ commute, and let $a_2, b_2 \in G_2$ be as above. By definition, $[a_2, b_2] = \beta([a_1, b_1]) = 1$, hence $a_2 \wr b_2 = 1$. This, and the relations of $G_2 \wr G_2$, ensure that ϕ is a \tilde{B}_0 -pairing. Thus ϕ induces a homomorphism $\gamma : G_1 \wr G_1 \rightarrow G_2 \wr G_2$ such that $\gamma(a_1 \wr b_1) = a_2 \wr b_2$ for all $a_1, b_1 \in G_1$. By symmetry there exists a homomorphism $\delta : G_2 \wr G_2 \rightarrow G_1 \wr G_1$ defined via α^{-1} . It is straightforward to see that δ is the inverse of γ , hence γ is an isomorphism.

Let $\kappa_1 : G_1 \wr G_1 \rightarrow [G_1, G_1]$ and $\kappa_2 : G_2 \wr G_2 \rightarrow [G_2, G_2]$ be the commutator maps. Since $\beta\kappa_1(a_1 \wr b_1) = \beta([a_1, b_1]) = [a_2, b_2] = \kappa_2\gamma(a_1 \wr b_1)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{B}_0(G_1) & \longrightarrow & G_1 \wr G_1 & \xrightarrow{\kappa_1} & [G_1, G_1] \longrightarrow 0 \\
& & \downarrow \tilde{\gamma} & & \downarrow \gamma & & \downarrow \beta \\
0 & \longrightarrow & \tilde{B}_0(G_2) & \longrightarrow & G_2 \wr G_2 & \xrightarrow{\kappa_2} & [G_2, G_2] \longrightarrow 0
\end{array}$$

Here $\tilde{\gamma}$ is the restriction of γ to $\tilde{B}_0(G_1)$. Since β and γ are isomorphisms, so is $\tilde{\gamma}$. This concludes the proof.

REFERENCES

- [1] M. Artin, and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London. Math. Soc. (3) **25** (1972), 75–95.
- [2] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk SSSR Ser. Mat **51** (1987), no. 3, 485–516.
- [3] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.
- [4] A. Hoshi, M. Kang, and B. E. Kunyavskii, *Noether problem and unramified Brauer groups*, arXiv:1202.5812v1, 2012.
- [5] B. E. Kunyavskii, *The Bogomolov multiplier of finite simple groups*, Cohomological and geometric approaches to rationality problems, 209–217, Progr. Math., 282, Birkhäuser Boston, Inc., Boston, MA, 2010.
- [6] C. Miller, *The second homology of a group*, Proc. Amer. Math. Soc. **3** (1952), 588–595.
- [7] P. Moravec, *Unramified groups of finite and infinite groups*, Amer. J. Math., to appear.
- [8] P. Moravec, *Groups of order p^5 and their unramified Brauer groups*, submitted.
- [9] E. Noether, *Gleichungen mit vorgeschriebener Gruppe*, Math. Ann. **78** (1916), 221–229.
- [10] I. R. Šafarevič, *The Lüroth problem*, Proc. Steklov Inst. Math. **183** (1991), 241–246.
- [11] D. J. Saltman, *Noether’s problem over an algebraically closed field*, Invent. Math. **77** (1984), 71–84.
- [12] R. G. Swan, *Noether’s problem in Galois theory*, in ‘Emmy Noether in Bryn Mawr’, Springer-Verlag, Berlin, 1983.

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